Markscheme

May 2018

Discrete mathematics

Higher level

Paper 3

11 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets** eg **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) G has an Eulerian trail because it has (exactly) two vertices (B and F) of odd degree ***R1***

(ii) G does not have an Eulerian circuit because not all vertices are of even degree ***R1***

[2 marks]

(b) for example BAEBCEFCDF ***A1A1***

Note: Award ***A1*** for start/finish at B/F, ***A1*** for the middle vertices.

[2 marks]

(c) (i) to determine the shortest route (walk) around a weighted graph ***A1***
using each edge (at least once, returning to the starting vertex) ***A1***

Note: Correct terminology must be seen. Do not accept trail, path, cycle or circuit.

(ii) we require the Eulerian trail in (b), (weight $=65$) ***(M1)***
and the minimum walk FEB (15) ***A1***
for example BAEBCEFCDFEB ***A1***

Note: Accept EB added to the end or FE added to the start of their answer in (b) in particular for follow through.

(iii) total weight is $(65+15=)80$ ***A1***

[6 marks]

Total [10 marks]

2. (a) **EITHER**

if p is prime (and a is any integer) then $a^p \equiv a \pmod p$ ***A1A1***

Note: Award ***A1*** for p prime and ***A1*** for the congruence or for stating that $p \mid a^p - a$.

OR

if p is prime (and $a \not\equiv 0 \pmod p$) then $a^{p-1} \equiv 1 \pmod p$ ***A1A1***

Note: Award ***A1*** for p prime and ***A1*** for the congruence or for stating that $p \mid a^{p-1} - 1$.

Note: Condone use of equals sign provided $(\text{mod } p)$ is seen.

[2 marks]

(b) (i) multiplying both sides of the linear congruence by a^{p-2} ***(M1)***

$a^{p-1}x \equiv a^{p-2}b \pmod p$ ***A1***

as $a^{p-1} \equiv 1 \pmod p$ ***R1***

$x \equiv a^{p-2}b \pmod p$ ***AG***

continued…

Question 2 continued

(ii) $x \equiv 5^{11} \times 7 (\bmod 13)$ ***M1***

$\equiv 341796875 (\bmod 13)$ ***(A1)***

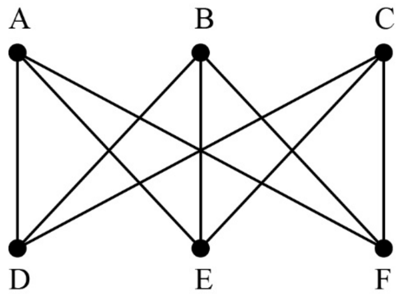
Note: Accept equivalent calculation *eg*, using $5^2 \equiv -1 \bmod 13$.

$\equiv 4 (\bmod 13)$ ***A1***

[6 marks]

Total [8 marks]

3. (a) (i)



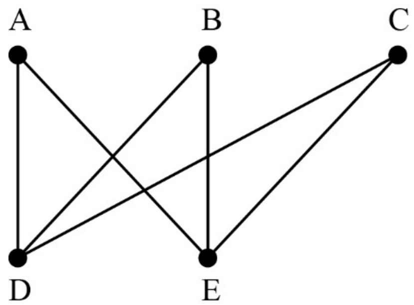
A1

(ii) for example ADBECFA ***A1***

Note: Accept drawing the cycle on their diagram.

Note: Accept Dirac's theorem (although it is not on the syllabus for (a)(ii). There is no converse that could be applied for (a)(iii).

(iii)



A1

a Hamiltonian cycle would have to alternate between the two vertex subsets which is impossible as $2 \neq 3$ ***R1***

Note: Award ***R1*** for an attempt to construct a Hamiltonian cycle and an explanation of why it fails, *eg*, ADBEC but there is no route from C to A without re-using D or E so no cycle. There are other proofs *eg*, have to go in and out of A, similarly B and C giving all edges leading to a contradiction.

[4 marks]

(b) (i) the sum of the vertex degrees is twice the number of edges ***A1***

continued…

Question 3 continued

(ii) assume G exists
the sum $2+3+3+4+4+5=21$ ***A1***
this is odd (not even) ***R1***
this contradicts the handshaking lemma
so G does not exist ***AG***

[3 marks]

(c) T has $v-1$ edges ***A1***

EITHER

if k vertices have degree 1 then $v-k$ vertices have degree ≥ 2 ***R1***
by the handshaking lemma
$2v-2 \geq 1 \times k + 2(v-k)(= 2v-k)$ ***M1***
this gives $k \geq 2$ ***A1***

OR

let S be the sum of vertex degrees
consider T having either no or one vertex of degree 1 ***R1***
case 1 suppose T has no vertices of degree 1 (*eg*, all vertices have degrees ≥ 2)
by the handshaking lemma
$S \geq 2v \neq 2(v-1)$ (not possible) ***A1***
case 2 suppose T has one vertex of degree 1 (*eg*, $v-1$ vertices have degrees ≥ 2)
by the handshaking lemma
$S \geq 2(v-1)+1 \neq 2(v-1)$ (not possible) ***A1***

THEN

so T has at least two vertices of degree 1 ***AG***

[4 marks]

Total [11 marks]

4. (a) **METHOD 1**

attempting to use the Euclidean algorithm ***M1***
$4k+2=1(3k+1)+(k+1)$ ***A1***
$3k+1=2(k+1)+(k-1)$ ***A1***
$k+1=(k-1)+2$ ***A1***
$=\gcd(k-1,\ 2)$ ***AG***

continued…

Question 4 continued

METHOD 2

$\gcd(4k+2, 3k+1)$

$= \gcd(4k+2-(3k+1), 3k+1)$ ***M1***

$= \gcd(3k+1, k+1) \ \left(= \gcd(k+1, 3k+1)\right)$ ***A1***

$= \gcd(3k+1-2(k+1), k+1) \ \left(= \gcd(k-1, k+1)\right)$ ***A1***

$= \gcd(k+1-(k-1), k-1) \ \left(= \gcd(2, k-1)\right)$ ***A1***

$= \gcd(k-1, 2)$ ***AG***

[4 marks]

(b) (i) (for k odd), $\gcd(4k+2, 3k+1) = 2$ ***A1***

(ii) (for k even), $\gcd(4k+2, 3k+1) = 1$ ***A1***

[2 marks]

Total [6 marks]

5. (a) attempt to find the auxiliary equation ($\lambda^2 - \lambda - 1 = 0$) ***M1***

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ ***(A1)***

the general solution is $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$ ***(M1)***

imposing initial conditions (substituting $n = 0,1$) ***M1***

$A + B = 0$ and $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$ ***A1***

$A = \frac{1}{\sqrt{5}}, \ B = -\frac{1}{\sqrt{5}}$ ***A1***

$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ ***A1***

[7 marks]

Note: Condone use of decimal numbers rather than exact answers.

continued…

Question 5 continued

(b) let $\text{P}(n)$ be $f_n > \alpha^{n-2}$ for integers $n \geq 3$

consideration of two consecutive values of f ***R1***

$f_3 = 2$ and $\alpha^{3-2} = \frac{1+\sqrt{5}}{2}(1.618\ldots) \Rightarrow \text{P}(3)$ is true ***A1***

$f_4 = 3$ and $\alpha^{4-2} = \frac{3+\sqrt{5}}{2}(2.618\ldots) \Rightarrow \text{P}(4)$ is true ***A1***

Note: Do not award ***A*** marks for values of n other than $n = 3$ and $n = 4$.

(for $k \geq 4$), assume that $\text{P}(k)$ and $\text{P}(k-1)$ are true ***M1***

required to prove that $\text{P}(k+1)$ is true

Note: Accept equivalent notation. Needs to start with 2 general consecutive integers and then prove for the next integer. This will affect the powers of the alphas.

$f_{k+1} = f_k + f_{k-1}$ (and $f_k > \alpha^{k-2}$, $f_{k-1} > \alpha^{k-3}$) ***M1***

$f_{k+1} > \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-3}(\alpha + 1)$ ***A1***

$= \alpha^{k-3}\alpha^2 = \alpha^{k-1} = \alpha^{(k+1)-2}$ ***A1***

as $\text{P}(3)$ and $\text{P}(4)$ are true, and $\text{P}(k)$, $\text{P}(k-1)$ true $\Rightarrow \text{P}(k+1)$ true

then $\text{P}(k)$ is true for $k \geq 3$ by strong induction ***R1***

Note: To obtain the final ***R1***, at least five of the previous marks must have been awarded.

[8 marks]

Total [15 marks]